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Evaluating Efficiencies of Turbofan Jet Engines: A Data Envelopment Analysis Approach

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Data envelopment analysis (DEA), a mathematical-programming-based method, has been developed in the operations research and economics literatures for effecting efficiency evaluations. Its use is explained and illustrated with an application to data on turbofan jet engines. Results are compared with standard engineering methods for measuring efficiency. In contrast to the latter, DEA handles data on multiple inputs and multiple outputs that can be used to characterize the performance of each engine. This is accomplished without recourse to any use of prearranged weights. DEA also provides information on the sources and amounts of inefficiencies in each input and output of each engine without requiring knowledge of the functional relations that give rise to these estimates. DEA represents an alternative method for evaluating efficiencies. Its purpose is to augment, not replace, traditional methods.

Nomenclature

A	= projected area of turbofan
$d(A, B)$	= distance from A to B
$\boldsymbol{e}$	= vector with all elements unity
$\dot{Q}_o$	= heat input to engine from fuel
$\boldsymbol{s}$	= vector of slack variables
T	= transpose
T_o	= cruise thrust
$\boldsymbol{u}$	= vector of weights (multipliers) to be determined for input
V	= velocity
V_o	= cruise velocity of plane
$\boldsymbol{v}$	= vector of weights (multipliers) to be determined for output
$\boldsymbol{x}$	= vector of inputs
$\boldsymbol{x}_o, \boldsymbol{y}_o$	= input and output vectors to be evaluated
$\boldsymbol{y}$	= vector of outputs
ε	= a positive (non-Archimedean infinitesimal element) smaller than any positive real number
η_0	= engineering efficiency of jet engine
θ	= scalar measure of efficiency
$\boldsymbol{\lambda}$	= intensity vector reflecting amount of each input and output used in an evaluation
$\boldsymbol{\theta}$	= vector of weights for outputs derived from $\boldsymbol{v}$
ρ	= density of air
ω	= vector of weights for inputs derived from $\boldsymbol{u}$

I. Introduction

THE typical engineering definition of efficiency for a jet engine is the ratio of work rate output over fuel energy input.[1] For average turbofan engines, this ratio is approximately 35%. In conformance with the usual engineering definition of efficiency, this evaluation refers only to a single input and a single output for each engine. Extensions to multiple inputs and multiple outputs are also possible that involve recourse to weighting schemes. Such weighting schemes generally raise additional problems, however, and also fall short in their ability to identify the sources and the amounts of inefficiencies in the inputs and the outputs for each engine relative to all other engines.

In this paper, we turn to another technique, data envelopment analysis (DEA), as developed relatively recently in the operations research and economics literature.[2] After initial investigations by Charnes et al.,[3] DEA has been extended and applied in many different situations (see the bibliography in Ref. 2, which cites some 500 articles and publications). As we shall illustrate with an example involving turbofan jet engines, DEA provides an alternative measure of overall efficiency. It additionally relates each engine to all other engines in the data set in order to identify a set of most efficient engines and uses this as a basis for estimating sources and amounts of inefficiencies for each engine. En route to accomplishing this, it provides an optimal set of weights. This differs from ordinary weighting schemes in that the weights are not imposed. Instead, they are derived from the data. The weights also yield the best (or highest) possible efficiency values for each of the engines being evaluated. Finally, all efficient engines that are used to affect each of these evaluations are also identified.

For an illustrative example, we use 29 turbofan jet engines. In this paper, we first describe the DEA methods and models to be used and relate them to the engineering definitions of efficiency. Next, we apply the DEA method to a class of turbojet engines used in commercial aircraft. The input and the output criteria used in the definition of efficiency are described and applied to a total of 29 engines. The results and interpretations of the DEA evaluations are then presented and contrasted with traditional efficiency measures. Finally, we provide extensions for consideration in the use of DEA modeling for engineering applications.

II. Models and Definitions

A. Single Input–Output Situations

We begin with a definition of efficiency in a single input–output situation, which is common in engineering. Although the efficiency definition varies from one engineering field to another, the following captures the basic engineering consideration:

$$0 \le E_o = y_o / y_T = (y_o/x_o)/(y_T/x_o) \le 1 \tag{1}$$

where y_T is the maximum possible (or theoretical) output that can be obtained from using input x_o and y_o is the actual output obtained

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from using the same input x_o. We then use E_o as the efficiency rating for the device (or engine) o and note that $0 \le E_o \le 1$, as is customary in engineering, with $E_o = 1$ if and only if $y_o = y_T$.

We now introduce nonnegative variables (weights, or multipliers) u and v to obtain the same E_o as in Eq. (1) from a mathematical programming model that will help to explain the extension to multiple input–output situations described later. Consider the following:

$$g_o = \max_{v,u} \left(\frac{vy_o}{ux_o} \,\middle|\, \frac{vy_T}{ux_o} \le 1, \frac{vy_o}{ux_o} \le 1, u, v \ge 0 \right) \tag{2}$$

The g_o value is maximal with $v^* y_T / u^* x_o \le 1$ and $v^* y_o / u^* x_o \le 1$, as represented in the constraints, where the superscript $*$ designates an optimal solution. Because $y_o \le y_T$, we have $E_o = g_o$ with $v^* y_o / u^* x_o \le v^* y_T / u^* x_o$, and hence $y_o / y_T \le x_o / x_o = 1$, as before. We thus have derived a customary engineering definition from a principle of optimality.

Now we generalize this to the case in which there are n devices to be evaluated. Let $J = \{j\}_{j=1}^n$ be the index set of these devices. Each device j produces output y_j by using input x_j for all $j \in J$. Let $j_o = o \in J$ be a device to be evaluated with output y_o by using x_o. If a known relationship gives an explicit function such as $f : x_o \longrightarrow y_T$, as constructed from theoretical derivations and/or experimental observations, one could rate each device relative to its theoretical maximum. In many situations, however, this function is unknown, or only partially known, so we want to avoid this assumption here.

To move toward this objective we utilize a path that will measure the relative efficiency of each device on the basis of only the input–output data observed from the devices being analyzed. This is accomplished by means of the following:

$$vy_T / ux_o = \max_{j \in J} (vy_j / ux_j) \tag{3}$$

which implies that the maximal value among the ratios of various outputs to various inputs observed from all devices is regarded as the maximum possible rating of any device o.

Extending Eq. (2) to accommodate Eq. (3), we have a model that can be used to measure the relative efficiency of each device, as follows:

$$h_o = \max_{v,u} \left(\frac{vy_o}{ux_o} \,\middle|\, \frac{vy_j}{ux_j} \le 1 \quad \forall j,\ u, v \ge 0 \right) \tag{4}$$

Because the device o being evaluated in the objective is also part of the constraints, it follows that $0 \le h_o \le 1$. Like E_o in Eq. (1), for any device o, we have a measure that can be applied successively to each device with $h_o = 1$ if the device o has full efficiency. A value of $h_o < 1$ means that 1) efficiency has not been achieved and 2) $1 - h_o$ measures the relative efficiency shortfall (or inefficiency) of the device being rated.

B. Extension to Multiple Input–Output Situations

We have provided a process for deriving a mathematical programming model as in Eq. (4) to evaluate the efficiency of performances that we developed from the engineering definition of efficiency essential in a single input–output situation. In fact, Eq. (4) is a special case of DEA, which uses an approach to treat multiple input–multiple output extensions of the above development.

To continue with the case of multiple inputs and multiple outputs, assume there are m inputs that each of n devices utilizes in order to produce s outputs. Let $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ be column vectors with components that respectively record the amounts of each of the m inputs and the s outputs for every one of the $j = 1, \ldots, n$ devices to be evaluated. As in Eq. (4), we introduce the weights $\boldsymbol{u}$ and $\boldsymbol{v}$ associated with inputs and outputs that are now column vectors of lengths m and s, respectively. Then we utilize the following model to evaluate device o with data $(\boldsymbol{x}_o, \boldsymbol{y}_o)$:

$$z_o = \max_{\boldsymbol{v},\boldsymbol{u}} \left(\boldsymbol{v}^T \boldsymbol{y}_o \big/ \boldsymbol{u}^T \boldsymbol{x}_o \right)$$

subject to

$$\boldsymbol{v}^T \boldsymbol{y}_j \big/ \boldsymbol{u}^T \boldsymbol{x}_j \le 1 \quad \forall j, \qquad \left(\boldsymbol{u}^T \boldsymbol{x}_o\right)^{-1} \cdot \boldsymbol{u}^T \ge \varepsilon \boldsymbol{e}^T$$
$$\left(\boldsymbol{u}^T \boldsymbol{x}_o\right)^{-1} \cdot \boldsymbol{v}^T \ge \varepsilon \boldsymbol{e}^T \tag{5}$$

To see how this relates to the preceding development, we can regard $\boldsymbol{u}^{*T} \boldsymbol{x}_o = \hat{x}_o$ as a virtual input and $\boldsymbol{v}^{*T} \boldsymbol{y}_o = \hat{y}_o$ as a virtual output. Given the optimum solution in terms of weights $(\boldsymbol{u}^*, \boldsymbol{v}^*)$ we can obtain a ratio z_o from the scalars $\hat{y}_o / \hat{x}_o$, in analogy to E_o in Eq. (1) and g_o in Eq. (2) and thereby identify this development with the ordinary engineering definitions of efficiency. Note that there is no weight to be specified in advance. Taking into account the performances of all devices, we secure the efficiency evaluation by choosing the best (or optimal) weights for the device being rated in the sense that these choices yield the highest possible z_o value that the data allow for each device. Thus, as different devices are brought into the objective, the same procedures will generally lead to choices of $\boldsymbol{u}^*$, $\boldsymbol{v}^*$ that differ for each of the evaluated devices. In all cases, $0 \le z_o \le 1$, because the data for the device being rated are also retained in the constraints.

It is assumed that all inputs and outputs to be weighted (before being summed) have some positive worth. It is, however, not necessary to specify these values. This is accomplished by introducing the non-Archimedean infinitesimal $\varepsilon > 0$ defined to be smaller than any positive real number. The constraints $(\boldsymbol{u}^T \boldsymbol{x}_o)^{-1} \cdot \boldsymbol{u}^T \ge \varepsilon \boldsymbol{e}^T$ and $(\boldsymbol{u}^T \boldsymbol{x}_o)^{-1} \cdot \boldsymbol{v}^T \ge \varepsilon \boldsymbol{e}^T$ provide the necessary mathematical formalism, where the $\boldsymbol{e}^T$ are row vectors with all components equal to unity but with possibly different lengths as determined from the contexts of their use. Practically, in most DEA computer codes the need for according a specific value to $\varepsilon > 0$ is avoided in terms of a two-phase operation analogous to the way the so-called big M is handled when a set of artificial variables is treated in ordinary linear programming. See Arnold et al.[4] for an extended use of the non-Archimedean element and detailed discussions of two-phase procedures in DEA and in linear programming.

C. Computational Aspects and Efficiency

Reference to Eq. (5) shows that it is a nonlinear (nonconvex) programming problem and hence is best used for conceptual clarification. However, this is an ordinary fractional programming problem. Hence it can be replaced with the linear programming equivalent by means of the following change of variables[5]:

$$t = \left(\boldsymbol{u}^T \boldsymbol{x}_o\right)^{-1}, \qquad \boldsymbol{\omega}^T = t\boldsymbol{u}^T, \qquad \boldsymbol{\theta}^T = t\boldsymbol{v}^T \tag{6}$$

Carrying out these substitutions in Eq. (5) we obtain

$$z_o = \max \boldsymbol{\theta}^T \boldsymbol{y}_o$$

subject to

$$\begin{aligned} \boldsymbol{\theta}^T \boldsymbol{y}_j - \boldsymbol{\omega}^T \boldsymbol{x}_j &\le 0 \quad \forall j \\ \boldsymbol{\omega}^T \boldsymbol{x}_o &= 1 \\ \boldsymbol{\omega}^T &\ge \varepsilon \boldsymbol{e}^T \\ \boldsymbol{\theta}^T &\ge \varepsilon \boldsymbol{e}^T \end{aligned} \tag{7}$$

The linear programming dual to Eq. (7) is

$$\min \theta_o - \varepsilon \left(\boldsymbol{e}^T \boldsymbol{s}_o^- + \boldsymbol{e}^T \boldsymbol{s}_o^+ \right)$$

subject to

$$\begin{aligned} \sum_j \boldsymbol{x}_j \lambda_j + \boldsymbol{s}_o^- &= \theta_o \boldsymbol{x}_o \\ \sum_j \boldsymbol{y}_j \lambda_j - \boldsymbol{s}_o^+ &= \boldsymbol{y}_o \end{aligned} \tag{8}$$

with all variables being nonnegative, but with θ_o otherwise unconstrained, where $\boldsymbol{s}_o^-$ is an m-column slack vector of input excesses and $\boldsymbol{s}_o^+$ is an $\boldsymbol{s}$ vector of output slacks for device o. See Ref. 5, which initiated the field of fractional programming.

Since $z_o \le 1$ in Eq. (7), it follows that the condition for full (100%) DEA efficiency of any device o becomes $z_o = 1$. In expression (8) this condition becomes

$$\theta_o^* = 1, \qquad \boldsymbol{s}_o^{-*} = \boldsymbol{s}_o^{+*} = 0 \tag{9}$$

We note that, by means of the dual theorem of linear programming, $z_o^* = \theta_o^* - \varepsilon(\boldsymbol{e}^T \boldsymbol{s}_o^{-*} + \boldsymbol{e}^T \boldsymbol{s}_o^{+*})$. Also note that $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ is represented on the left-hand as well as the right-hand side of expression (8). Hence,

$\theta_o = 1$ is always achievable so $\theta_o^* \leq 1$. We thus conclude that $z_o = 1$ if and only if the conditions defined in Eqs. (9) hold. A proof and extended description can be found in Ref. 6, p. 67.

A value of $\theta_o^* < 1$ means that all inputs x_{io}, $i = 1, \ldots, m$ of device o can be reduced to the levels $\theta_o^* x_{io}$ without changing their proportions. In the DEA literature, this is referred to as "(pure) technical inefficiency" and is interpreted to mean that the evidence shows that this proportional reduction of all inputs could be accomplished without worsening any output. If, in addition, $s_{io}^{-*} > 0$ for any component of s_o^{-*}, then a still further reduction in input i can be made. This further reduction alters the input proportions and hence represents a mix inefficiency. Note that both of the pure technical and mix inefficiencies are to be comprehended by technical inefficiency.[7,8] In either or both cases, this means that the following improvements can be made:

$$\hat{x}_{io} = \theta_o^* x_{io} - s_{io}^{-*} \leq x_{io}, \qquad i = 1, \ldots, m \tag{10a}$$

Moreover, inspection of the structure of expression (8) shows that these improvements can be effected without worsening any other input and output. Similarly, if $s_{ro}^{+*} > 0$ for any component of s_o^{+*} then

$$\hat{y}_{ro} = y_{ro} + s_{ro}^{+*} \geq y_{ro}, \qquad r = 1, \ldots, s \tag{10b}$$

This means that the rth output can also be improved without worsening any other input and output.

This interpretation of Eqs. (9) leads to the following definition of efficiency.

D. Definition of Efficiency

A device is to be regarded as 100% efficient if and only if it is not possible to improve any of its inputs or outputs without worsening some of its other inputs or outputs.

The optimizations in Eq. (7) and expression (8) represent elements in models designed to implement this definition of efficiency. Thus, if Eqs. (9) are fulfilled, then there is no evidence of relative inefficiency. Specifically, this result means that no other device and no nonnegative combination (in a piecewise linear manner) of the performances exhibited by the other devices is able to improve on the performance exhibited by the device being evaluated in the objective of expression (8).

It is to be emphasized that here we are confining attention to the physical (or engineering) aspects of efficiency. In the DEA literature this is referred to as "technical efficiency" and distinguished from other types of efficiency such as "price" and "allocative efficiency" in which economic or other types of evaluation elements permit a worsening of some input or output in order to improve some other input or output.

E. Geometrical Portrayals

The situation we are considering may be idealized by reference to Fig. 1. Here six devices are indicated, each of which utilizes two inputs in amounts (x_1, x_2) to produce one unit of the same output. The performance of F utilizes more of both x_1 and x_2 than C and hence is clearly inefficient. Thus, using expression (8) would result in a value of $\theta_F^* < 1$. Graphically, this can be identified as the ratio of the distances from 0, the origin, to F′ and to F. In this sense the θ value is referred to as a radial measure of efficiency that we can formalize as

$$0 \leq \theta_F^* = \frac{d(0, F')}{d(0, F)} \leq 1 \tag{11}$$

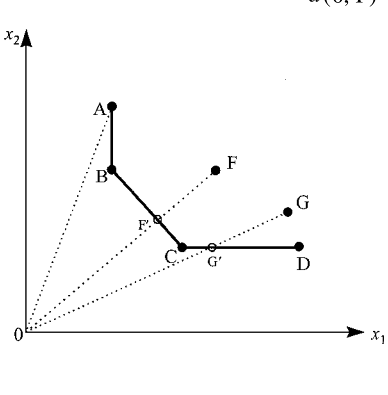

Fig. 1 Efficiency evaluations in two inputs and one output example.

with $d(0, F') \leq d(0, F)$, where d is a Euclidean (or L_2 metric) distance function. This signifies the proportional reduction in both inputs that could be reached by moving from F to the point F′ generated from a convex combination of B and C.

The situation for G is slightly more complex. Here we have

$$0 \leq \theta_G^* = \frac{d(0, G')}{d(0, G)} < 1 \tag{12}$$

with $d(0, G') < d(0, G)$, so both inputs of G could be reduced in the proportion $1 - \theta_G^*$ to improve this performance without worsening its output. This movement from G to G′, however, does not imply an end to the efficiency improvement of G. Further improvement can be identified from a comparison of G′ and C. Obviously C uses less x_1 and no more x_2 than G′ in production of the same one-unit output. Thus the further improvement in moving from G′ to C represents a removal of the nonzero slack s_{1G}^{-*}.

The situation at A is different in that $\theta_A^* = 1$, which satisfies one of the conditions for efficiency in Eqs. (9). However, the amount of x_2 utilized by A exceeds its corresponding value in B, which therefore produces the same one unit of output with less x_2 and no more x_1. This difference in the x_2 values is represented by s_{2A}^{-*}, the removal of which would position A at B.

As can be seen, all evaluations are effected relative to the frontier represented by the solid lines connecting A to D in a piecewise linear manner. Movement from A to B or from D to C may be effected by removing slacks that will improve one input without worsening the other input. Only between B and C is such movement not possible. This represents the efficient frontier portion of the frontier. It is from this portion of the frontier that all of the efficiency evaluations are made.

To put this all together, we move to the general situation by extending Eqs. (9) to the following expressions:

$$\hat{\boldsymbol{x}}_o = \theta_o^* \boldsymbol{x}_o - \boldsymbol{s}_o^{-*} = \sum_j \boldsymbol{x}_j \lambda_j^*, \qquad \hat{\boldsymbol{y}}_o = \boldsymbol{y}_o + \boldsymbol{s}_o^{+*} = \sum_j \boldsymbol{y}_j \lambda_j^* \tag{13}$$

which project the originally observed data $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ into $(\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o)$ As shown in the graphical interpretation above, the values $\hat{\boldsymbol{x}}_o$ and $\hat{\boldsymbol{y}}_o$ become the coordinates of the point on the efficient frontier used to effect the evaluation of device o.

The expressions on the left-hand side of Eqs. (13), referred to as the Charnes, Cooper, and Rhodes[3] (CCR) projection formula, specify all inefficiencies that can be identified in the performance of device o without any worsening of inputs and outputs. Charnes et al.[3] first introduced Eqs. (13) and analyzed their properties. The computer codes used in DEA generally use variants of the simplex method that utilize basis sets to affect the desired optimum solutions. Hence we can replace Eqs. (13) with

$$\hat{\boldsymbol{x}}_o = \theta_o^* \boldsymbol{x}_o - \boldsymbol{s}_o^{-*} = \sum_{j \in J_B} \boldsymbol{x}_j \lambda_j^*, \qquad \hat{\boldsymbol{y}}_o = \boldsymbol{y}_o + \boldsymbol{s}_o^{+*} = \sum_{j \in J_B} \boldsymbol{y}_j \lambda_j^* \tag{14}$$

where J_B represents the basis index set when an optimal solution is obtained in evaluating device o so that $j \in J_B$ and DMU_j is also efficient if $\lambda_j^* > 0$.

To summarize, we have a set of efficient devices with the vectors $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ $\forall j \in J_B$, which measures the efficiency of the device with coordinates $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ being evaluated. The coordinates $\hat{\boldsymbol{x}}_o, \hat{\boldsymbol{y}}_o$ correspond to points on the efficient frontier generated by the right-hand side expressions in Eqs. (14). The set J_B therefore comprises the peer group that is used to evaluate device o. This identification can be of value by directing attention to members of the peer group when information is sought on how to improve the performances in each of the particular inputs and outputs in $(\boldsymbol{x}_o, \boldsymbol{y}_o)$. The Public Utility Commissions of Texas have made extensive use of these peer groups in the efficiency audits of utilities that they are required to perform under legislative mandates.[9] Hence we can use Fig. 1 to illustrate this point by noting that B and C enter with positive values of λ_B^* and λ_C^* in the evaluation of F. Similarly, either B or C or both will have a positive lambda (λ) value as a member of the basis sets used to evaluate all of the other points in Fig. 1 as well.

III. Turbofan Jet Engine Data

A. Input–Output Criteria

We have chosen to focus our study on data for 29 turbofan jet engines being used on commercial airplanes. The planes on which the engines are used vary in size and flight pattern (i.e. altitude, cruise velocity, length of flight, etc.), but these characteristics are not used directly to determine the efficiency of the jet engine. The data were collected from several sources[1,10,11] and a tabulated version is presented in Table 1. A total of three inputs and two outputs, described below, were chosen to illustrate the DEA method. Although this study is to serve only as an example of possible applications of DEA to engineering situations, the inputs and outputs were chosen to represent useful engineering parameters. Other inputs such as cost, engine maintenance, etc. could also be included.

B. Inputs

Fuel Consumption (in Pounds Mass per Hour)

The fuel consumption input was backed out of the thrust-specific fuel consumption and cruise thrust data for each engine. It was separated to allow it to be identified as an input in its own right.

Weight of Engine (in Pounds Force)

The weight of the engine is important mainly when the engine is used on smaller aircraft. As aircraft get larger and are set up for longer flights, the weight of the fuel supersedes the weight of the engine as a concern. For these larger aircraft, the fuel consumption becomes the most important factor. Weight was also chosen to illustrate how DEA can utilize nondynamic parameters for an efficiency rating. We picked this parameter because it could have ramifications on the dynamic performance of the engine when applied to an airplane.

Drag (in Pounds Force)

The drag is a relative force that is a function of fan diameter (area), cruise velocity, and air density at cruise altitude. For this study, we chose a drag equal to 1% of the force caused by stagnation pressure on the projected area of the fan. The formula is

$$\text{drag} = 1\% \text{ of } \left(\tfrac{1}{2}\rho V^2 A\right)$$

where ρ is the density of air at altitude of velocity data, V is the velocity of a plane with a given engine at reference altitude (calculated from the Mach number), and A is the projected area of the fan surface into flow (a circle). Table A1 in Appendix A lists the values of these parameters for all 29 engines, which data are used to calculate the drag values under inputs in Table 1.

C. Outputs

Airflow (in Pounds Mass per Second)

The airflow is important for the propulsive efficiency of a jet engine. Propulsive efficiency increases as the airflow increases and/or exit velocity decreases. The limiting factors on airflow are in the fan (diameter and geometry) and with the number of compressor stages.

Cruise Thrust (in Pounds Force)

Cruise thrust allows for a comparison of the propulsive efficiency of the engine. Cruise velocity is not used because it represents the drag characteristics (or design) of the plane.

There are some implicit relationships between the inputs and the outputs that add explanatory power to the results. Direct engineering relationships were not used because they contain many predefined ratios (such as the bypass ratio). Using ratios creates a loss of information in that, for example, 10% cannot distinguish between $\frac{1}{10}$ and $\frac{20}{200}$. DEA allows for data to be separated into their core form and is not dependent on the consistency of units (e.g., pounds, feet, dollars, etc.) that may therefore differ for the different inputs and different outputs. Although DEA can handle many more inputs and outputs, we limited the number of inputs to three and the number of outputs to two in order to simplify the presentation. This gave adequate degrees of freedom for the study without overcomplicating the illustration.

IV. Results and Investigations

A. Efficiency Classification

We now apply DEA and run the model given by expression (8) to evaluate the efficiency of these engines by using the data in Table 1. Table 2 shows the results for the 29 engines, which include the radial efficiency measures (θ_o^*) and the slacks of inputs and outputs.

Table 1 Data for 29 turbofan jet engines

Engines			Inputs			Outputs	
Make	Model	Number	Fuel consumption, lbm/hr	Weight, lbf	Drag, lbf	Airflow, lbm/s	Cruise thrust, lbf
AlliedSignal	TFE731-2	1	615	743	246	113	755
AlliedSignal	TFE731-3	2	682	754	246	118	817
AlliedSignal	TFE731-5B	3	795	899	246	143	1,052
AlliedSignal	TFE731-20	4	638	836	246	123	876
CFE	738	5	937	1,325	571	210	1,464
CFM	CFM56-2B	6	3,265	4,671	1,831	817	4,969
CFM	CFM56-3	7	3,262	4,301	1,595	655	4,890
CFM	CFM56-3C	8	3,200	4,301	1,413	688	4,885
CFM	CFM56-5A	9	2,980	4,975	1,831	852	5,000
CFM	CFM56-5C	10	3,311	5,494	2,052	1,027	5,840
CFM	CFM56-5CX	11	3,597	5,494	2,770	1,027	6,600
Garrett	TFE731-5	12	760	850	387	140	986
GE	GE90-76B	13	9,100	16,664	7,049	3,000	17,500
GE	GE90-76B	14	9,100	16,664	7,049	3,120	17,500
GE	CF6-50C2	15	7,280	8,490	2,930	1,476	11,555
GE	CF6-80C2	16	6,912	9,135	3,395	1,650	12,000
IAE	V2535	17	3,307	5,224	1,558	848	5,752
IAE	V2528-D5	18	3,314	5,511	1,321	825	5,773
P&W	JT8D-15A	19	3,833	3,474	1,197	327	4,920
P&W	JT9D-59A	20	7,720	9,140	4,135	1,639	11,950
P&W	PW2037	21	3,783	7,300	3,187	1,210	6,500
P&W Canada	PW300	22	751	993	411	180	1,113
Rolls Royce	535C	23	5,461	7,294	2,144	1,142	8,453
Rolls Royce	535E4B	24	5,203	7,189	2,179	1,150	8,700
Rolls Royce	RB211-22B	25	5,963	9,195	3,187	1,380	9,495
Rolls Royce	RB211-524B	26	7,073	9,814	3,262	1,513	11,000
Rolls Royce	RB211-524H	27	6,733	9,874	3,300	1,604	11,813
Rolls Royce	RB211-535E	28	5,156	7,306	2,179	1,151	8,495
Rolls Royce	Trent895	29	7,241	13,133	5,113	2,700	13,000

Table 2 DEA efficiency results

Engines			Radial efficiency (θ_0^*)	Input slacks			Outputs slacks	
Make	Model	Number		Fuel consumption	Weight	Drag	Airflow	Cruise thrust
Allied signal	TFE731-2	1	0.8575	29.96	0	0	0	0
Allied signal	TFE731-3	2	0.8961	67.59	0	0	0	0
Allied signal	TFE731-5B	3[a]	1	0	0	0	0	0
Allied signal	TFE731-20	4	0.8937	0	0	0	0	0
CFE	738	5	0.8780	0	0	0	3.34	0
CFM	CFM56-2B	6	0.9114	149.77	0	52.58	0	0
CFM	CFM56-3	7	0.8652	0	0	0	16.40	0
CFM	CFM56-3C	8	0.9238	0	0	0	0	0
CFM	CFM56-5A	9	0.9285	0	0	0	0	0
CFM	CFM56-5C	10	0.9838	0	0	0	0	0
CFM	CFM56-5CX	11[a]	1	0	0	0	0	0
Garrett	TFE731-5	12	0.9015	118.41	0	62.97	0	0
GE	GE90-76B	13	1	0	0	0	120	0
GE	GE90-76B	14[a]	1	0	0	0	0	0
GE	CF6-50C2	15[a]	1	0	0	0	0	0
GE	CF6-80C2	16[a]	1	0	0	0	0	0
IAE	V2535	17	0.9898	0	0	0	0	0
IAE	V2528-D5	18[a]	1	0	0	0	0	0
P&W	JT8D-15A	19[a]	1	0	0	0	0	0
P&W	JT9D-59A	20	0.9941	768.14	0	742.00	0	0
P&W	PW2037	21	0.9105	0	356.15	300.01	0	0
P&W Canada	PW300	22	0.9496	79.33	0	32.98	0	0
Rolls Royce	535C	23	0.9628	0	0	0	0	0
Rolls Royce	535E4B	24	0.9986	0	0	0	29.90	0
Rolls Royce	RB211-22B	25	0.8928	0	0	0	41.70	0
Rolls Royce	RB211-524B	26	0.9022	0	0	0	4.49	0
Rolls Royce	RB211-524H	27	0.9998	0	0	0	86.50	0
Rolls Royce	RB211-535E	28	0.9749	0	0.14	0	11.90	0
Rolls Royce	Trent895	29[a]	1	0	0	0	0	0

[a]Eight efficient engines.

Our first finding is that 8 out of 29 engines are DEA efficient. We also find all zero slacks in the last column under cruise thrust. This provides a guide to show how each inefficient engine can be improved by reference to each input–output source without changing cruise thrusts where all slacks are zero.

As shown in Table 2, the engines found to be inefficient fall into three categories, as follows: 1) the radial efficiency score $\theta_o^* = 1$ but nonzero slacks are present (see engine 13), 2) $\theta_o^* < 1$ and the slacks are all zero (see, e.g., engine 4), and 3) $\theta_o^* < 1$ and nonzero slacks are revealed (see e.g., engine 1). For engine 1, this means that all three of the inputs (fuel consumption, weight, and drag) would need to be reduced by nearly 15% $(1 - \theta_o^*)$ to achieve purely technical efficiency. The fuel consumption input would also need to be reduced an additional 30 lb · m/h to achieve a full (100%) efficient rating.

It should be noted that these results are relative to the performances of the other engines. In particular, the rating of an engine is affected by a peer group chosen by means of the optimization used in DEA, as discussed in Eqs. (13) and (14). To see this, we turn to Tables 3 and 4, which represent a standard type of DEA report at the individual engine level. These examples are slightly modified versions of reports generated by a DEA code that is available from the Center for Management of Operations and Logistics at The University of Texas at Austin, Austin, Texas. Other computer codes are described and evaluated in Ref. 2.

At the top of Table 3 the theta value is, of course, the same as the 0.8575 rating shown for engine 1 in Table 2. The data in the column under "Value if efficient" of the first table are the CCR projected values by use of Eqs. (13), which means that engine 1 can be DEA efficient if it uses these values. Moving to the second part of Table 3, one can find the peer group members and their lambda values and also their actual observed data. The peer group of engine 1 consists of engines 3, 15, and 29, whose lambda values are 0.2633, 0.0259, and 0.0137, respectively. As noted in Eqs. (14), the actual data of these peer engines construct the efficient facet that affects all inefficiency scores and identifies sources and amounts of inefficiency for engine 1 being evaluated. A point we want to emphasize here is that engine 3 with its lambda value of 0.2633 gives a relatively stronger effect to the efficiency evaluation of engine 1 than the others in this peer group. In fact, engine 3 among these

Table 3 Efficiency summary for engine 1: Allied signal TFE731-2

Criteria	Value observed	Slack	Value if efficient
Radial efficiency score (theta) = 0.8575			
Fuel consumption	615	30	497
Weight	743	0	637
Drag	246	0	211
Airflow	113	0	113
Cruise thrust	755	0	755
Peer group (facet) and their actual data			
Engine number	3	15	29
Engine name	TFE731-5B	CF6-50C2	Trent895
Lambda value	0.2633	0.0259	0.0137
Fuel consumption	795	7,280	7,241
Weight	899	8,490	13,133
Drag	246	2,930	5,113
Airflow	143	1,476	2,700
Cruise thrust	1,052	11,555	13,000

Table 4 Efficiency summary for engine 13: GE GE90-76B

Criteria	Value observed	Slack	Value if efficient
Radial efficiency score (theta) = 1.0000			
Fuel consumption	9,100	0	9,100
Weight	16,664	0	16,664
Drag	7,049	0	7,049
Airflow	3,000	120	3,120
Cruise thrust	17,500	0	17,500
Peer group (facet) and their actual data			
Engine number	14	—	—
Engine name	GE90-76B	—	—
Lambda value	1.000	—	—
Fuel consumption	9,100	—	—
Weight	16,664	—	—
Drag	7,049	—	—
Airflow	3,120	—	—
Cruise thrust	17,500	—	—

peers has the most similar size of actual input–output data to that of engine 1. This is a frequent result in which DEA suggests such a predominant device as a benchmark for the device under evaluation.

Table 4 notes another situation in which $\theta_o^* = 1$ for engine 13 but efficiency is not achieved because of nonzero slack in airflow. The data of engine 13 are the same as those of engine 14 except for airflow output values (i.e., airflow output of engine 13 = 3000, but for engine 14 = 3120). Thus the peer group for engine 13 consists only of engine 14, so that, as found from our DEA analysis, the data show that engine 13 must increase its airflow output by the slack amount of 120 (= 3120 − 3000) in order to be efficient.

All information on DEA efficiency, including the peer groups used to evaluate each of the 29 engines, are shown in Table A2 of Appendix A. In fact, Tables 3 and 4 simply recapitulate these results in a different form for engines 1 and 13. Thus the two tables may be used in tandem, if this is needed, and Table A2 may then be used to guide attention when the recapitulation is wanted.

Witness, for instance, the case of engine 20 in Table A2. The value of $\theta_o^* = 0.9941$ might tempt one to regard the deviation from unity as trivial. However, the slack values are far from optimal and, moreover, they have the largest of all values in their respective inefficiency characterizations. Further inquiry might therefore be warranted. Directing attention to the peer group column, we find that the largest lambda value of the peer group members is $\lambda_{16}^* = 0.9606$, which is associated with engine 16 (this engine is efficient because it has a positive coefficient). A printout comparison in the manner of Table 4 would show engines 16 and 20 to be very close in their outputs and in the weight input. The other two inputs, however, show very large inefficiencies for engine 20 compared with those of 16.

B. Comparison with Engineering Efficiency

By using the standard engineering definition for calculating the efficiency of a jet engine, we can obtain a comparison between DEA and engineering efficiency. The formula used for calculating the traditional engineering efficiency is

$$\eta_o = T_o V_o / \dot{Q}_o \tag{15}$$

where T_o is the thrust of the engine (taken at cruise), V_o is the cruise velocity of the jet, and $\dot{Q}_o$ is the heat input from fuel (mass flow rate of fuel times low heating value).[1] The low heating value for all engines was calculated with JP-4 jet fuel.

The engineering efficiencies are calculated and presented in the middle of Table 5 for all 29 engines. Table 5 also shows the DEA radial efficiency scores (theta), as in Table 2, to compare these two efficiency results. The DEA study indicated that engines 3 and 19 were efficient, but the engineering efficiency is low.

We should note that the use of different input–output criteria in DEA affects the difference (or discrepancy) between the DEA and engineering efficiencies. When the input–output criteria for DEA is the same as the traditional engineering efficiency measure given by

Table 5 Engineering efficiency data and comparison with DEA theta values

Parameter values			Engineering efficiency		DEA vs engineering efficiency		
T (N)	V (m/s)	$\dot{Q}$ (W)	η (%)	Engine number	DEA efficiency (θ)	Engine in order	Engineering efficiency η (%)
3,358	236	3,318,201	23.88	1	1	14	37.63
3,634	236	3,678,804	23.31	2	1	13	37.63
4,680	236	4,288,798	25.75	3	1	29	36.45
3,897	236	3,439,011	26.74	4	1	11	35.90
6,512	236	5,052,649	30.41	5	1	18	34.09
22,103	237	17,604,855	29.78	6	1	16	33.97
21,752	252	17,588,661	31.17	7	1	15	31.06
21,729	237	17,254,562	29.87	8	1	3	25.75
22,241	237	16,069,943	32.83	9	1	19	25.68
25,977	237	17,856,403	34.51	10	0.9998	27	36.47
29,358	237	19,397,176	35.90	11	0.9986	24	32.72
4,386	236	4,099,486	25.24	12	0.9941	20	32.18
77,844	237	49,072,645	37.63	13	0.9898	17	34.03
77,844	237	49,072,645	37.63	14	0.9838	10	34.51
51,399	237	39,256,229	31.06	15	0.9749	28	32.24
53,378	237	37,273,640	33.97	16	0.9628	23	30.29
25,586	237	17,835,480	34.03	17	0.9496	22	28.84
25,679	237	17,869,464	34.09	18	0.9285	9	32.83
21,885	242	20,668,104	25.68	19	0.9238	8	29.87
53,156	252	41,629,242	32.18	20	0.9114	6	29.78
28,913	252	20,400,200	35.72	21	0.9105	21	35.72
4,951	236	4,051,324	28.84	22	0.9022	26	32.33
37,601	237	29,447,028	30.29	23	0.9015	12	25.24
38,699	237	28,055,532	32.72	24	0.8961	2	23.31
42,236	252	32,155,309	33.10	25	0.8937	4	26.74
48,930	252	38,141,848	32.33	26	0.8928	25	33.10
52,547	252	36,310,576	36.47	27	0.8780	5	30.41
37,787	237	27,806,745	32.24	28	0.8652	7	31.17
57,827	246	39,047,805	36.45	29	0.8575	1	23.88

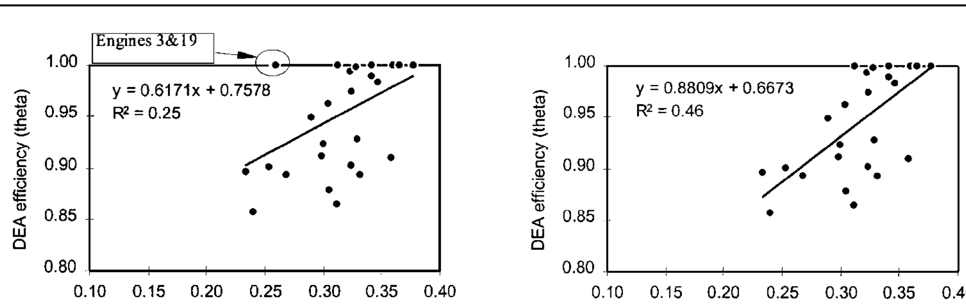

Fig. 2 Comparative portrayal by means of regression analysis for a) all 29 engine data and b) 27 engine data, except the data of engines 3 and 19.

Eq. (5), the relative ranking of all engines is the same as expected. DEA considers weight of the engine as an input and airflow as an output, and these do not show up in the engineering approach to overall efficiency. This is clear if engines 13 and 14 are compared, as is done in Table 4. The engineering efficiencies are exactly the same, which would be expected if airflow were not considered. Note that nonzero slack for airflow in Table 4 is the only parameter that differs between engines 13 and 14, and this relative inefficiency in performance is not reflected in the engineering measure used.

To provide a different perspective, we turn to the scatter diagrams and regressions exhibited in Fig. 2, where x, the explanatory variable represents engineering efficiency (η) and y, the dependent variable, represents the DEA evaluation (θ^*). Figure 2a suggests that two engines, 3 and 19, might be characterized as outliers.

To check this possibility we return to Table A2 in Appendix A. Moving to the peer group columns, we note that engines 3 and 19 are designated as self-evaluators (in DEA terminology). This characterization can have a variety of causes. It can, for instance, mean that special features are present that set such a self-evaluator apart from all of the rest. Alternatively it can mean that only the (one or more) special features of the engine being evaluated are the cause of this achieved score, although other aspects of its performance are poor. The engineering efficiencies calculated for engines 3 and 19 suggest that this may be the case here. In any event, DEA has identified each of engines 3 and 19 as being evaluated by itself, viz., it is the only member of its own peer group.

This suggests that other analytical tools might well be brought to bear in order to ensure that relative inefficiencies are not being concealed in this manner. Further illustration is given in Appendix B with an analytical tool, or envelopment map. See, e.g., Ref. 12 for a use of these tools in association with the use of DEA to evaluate maintenance activities in the U.S. Air Force Fighter Command. Here we simply treat this as justification for regarding engines 3 and 19 as outliers and therefore remove these observations. The resulting new regression is shown in Fig. 2b.

Evidently this removal improves the regression fit as measured by R^2. Moreover, all parameter estimates are statistically significant in both regressions, and the increase in R^2, going from Fig. 2a to Fig. 2b, is also significant. Second, the t values associated with slopes 0.6171 and 0.8809 increases from $t = 2.949$ (with $p = 0.0034$) to $t = 4.553$ (with $p \approx 0.000^+$).

Although statistical significance is achieved, both $R^2 < 0.5$, which means that the explanatory power is low. To remedy this it would be necessary to add more variables to these regressions if the engineering and the DEA results are to be brought closer together. However, it is not clear how to augment the single output–single input engineering definition. On the other hand, there is no reason why more variables cannot be added to the DEA analysis, and this would seem to be the way to go, rather than trying to reduce the number of variables used in the DEA analysis (in order to bring it closer to the engineering definition).

V. Conclusions

In this example, we gave attention to only technical efficiencies of jet engines so that we could effect comparisons between DEA and engineering definitions of efficiency. This kind of comparison can be extended to include economic and/or other considerations. In DEA this would extend the model to allow for tradeoffs between some inputs or outputs in which some could be worsened in favor of improving other inputs or outputs when there is an economic advantage. This extension is referred to as allocative efficiency in DEA and, as described by Charnes et al.,[13] extensions can be made to describe yet other types of efficiency.

When the parameters chosen for the DEA are the same as those used to determine the engineering efficiency, the results (i.e., relative ranking) are the same. However, this study was intended to show how other parameters (such as weight, engine drag, and fuel consumption) could be accounted for in producing an alternative efficiency rating of a jet engine.

Another added value in using DEA is that it returns not only an overall inefficiency parameter value, but also the amount of the inefficiency in each input and output. In addition, it identifies the peer group from which these evaluations are made. It should, however, be noted that the efficiency rating returned by DEA is a relative rating, as obtained by a comparison of each engine with all other engines by use of the same inputs and outputs. As is commonly

Appendix A: Summary Tables for Secondary Data and DEA Results

Table A1 Secondary data used to calculate drag input in Table 1

Engine number	TSFC, (lb · m/h)/ft · lb	Mach number	Altitude, kft	a/a_{ref}[a]	a, ft/s	Cruise velocity, ft/s	Diameter, in.	Density (ratio)	Density,[b] lbm/ft^3
1	0.815	0.80	40	0.8671	967.68	774.15	28.2	0.2471	0.018895737
2	0.835	0.80	40	0.8671	967.68	774.15	28.2	0.2471	0.018895737
3	0.756	0.80	40	0.8671	967.68	774.15	28.2	0.2471	0.018895737
4	0.728	0.80	40	0.8671	967.68	774.15	28.2	0.2471	0.018895737
5	0.640	0.80	40	0.8671	967.68	774.15	43.0	0.2471	0.018895737
6	0.657	0.80	35	0.8717	972.82	778.25	68.3	0.3108	0.023766876
7	0.667	0.85	35	0.8717	972.82	826.89	60.0	0.3108	0.023766876
8	0.655	0.80	35	0.8717	972.82	778.25	60.0	0.3108	0.023766876
9	0.596	0.80	35	0.8717	972.82	778.25	68.3	0.3108	0.023766876
10	0.567	0.80	35	0.8717	972.82	778.25	72.3	0.3108	0.023766876
11	0.545	0.80	35	0.8717	972.82	778.25	84.0	0.3108	0.023766876
12	0.771	0.80	40	0.8671	967.68	774.15	35.4	0.2471	0.018895737
13	0.520	0.80	35	0.8717	972.82	778.25	134.0	0.3108	0.023766876
14	0.520	0.80	35	0.8717	972.82	778.25	134.0	0.3108	0.023766876
15	0.630	0.80	35	0.8717	972.82	778.25	86.4	0.3108	0.023766876
16	0.576	0.80	35	0.8717	972.82	778.25	93.0	0.3108	0.023766876
17	0.575	0.80	35	0.8717	972.82	778.25	63.0	0.3108	0.023766876
18	0.574	0.80	35	0.8717	972.82	778.25	58.0	0.3108	0.023766876
19	0.779	0.80	30	0.8911	994.47	795.57	49.2	0.3747	0.028653309
20	0.646	0.85	35	0.8717	972.82	826.89	96.6	0.3108	0.023766876
21	0.582	0.85	35	0.8717	972.82	826.89	84.8	0.3108	0.023766876
22	0.675	0.80	40	0.8671	967.68	774.15	36.5	0.2471	0.018895737
23	0.646	0.80	35	0.8717	972.82	778.25	73.9	0.3108	0.023766876
24	0.598	0.80	35	0.8717	972.82	778.25	74.5	0.3108	0.023766876
25	0.628	0.85	35	0.8717	972.82	826.89	84.8	0.3108	0.023766876
26	0.643	0.85	35	0.8717	972.82	826.89	85.8	0.3108	0.023766876
27	0.570	0.85	35	0.8717	972.82	826.89	86.3	0.3108	0.023766876
28	0.607	0.80	35	0.8717	972.82	778.25	74.5	0.3108	0.023766876
29	0.557	0.83	35	0.8717	972.82	807.44	110.0	0.3108	0.023766876

[a] $a_{ref} = 1116$ ft/s. [b] $\rho_{ref} = 0.07647$ lbm/ft^3.

Table A2 DEA results for 29 turbofan jet engines

Engine number	Input[b]			Output[b]		Radial theta	Input slack			Output slack		Peer group[c] Lambda			
	Fuel consumption	Weight	Drag	Airflow	Cruise thrust		Fuel consumption	Weight	Drag	Airflow	Cruise thrust				
1	615	743	246	113	755	0.8575	29.96	0	0	0	0	#3	#15	#29	
	497.40	637.12	210.95	113	755							0.2633	0.0259	0.0137	
2	682	754	246	118	817	0.8961	67.59	0	0	0	0	#3	#15	#29	
	543.55	675.66	220.44	118	817							0.2947	0.0319	0.0107	
3[a]	795	899	246	143	1,052	1.0000	0	0	0	0	0	Itself			
	795	899	246	143	1,052										
4	638	836	246	123	876	0.8937	0	0	0	0	0	#3	#15	#18	#29
	570.17	747.12	219.85	123	876							0.2727	0.0252	0.0403	0.0050
5	937	1,325	571	210	1,464	0.8780	0	0	0	3.34	0	#11	#16		
	822.71	1,163.39	501.36	213.34	1,464							0.0929	0.0686		
6	3,265	4,671	1,831	817	4,969	0.9114	149.77	0	52.58	0	0	#16	#29		
	2,825.95	4,257.15	1,616.19	817	4,969							0.2553	0.1466		
7	3,262	4,301	1,595	655	4,890	0.8652	0	0	0	16.40	0	#15	#16	#18	
	2,822.28	3,721.23	1,379.99	671.40	4,890							0.0090	0.3982	0.0013	
8	3,200	4,301	1,413	688	4,885	0.9238	0	0	0	0	0	#15	#16	#18	#29
	2,956.16	3,973.26	1,305.33	688	4,885							0.2480	0.0392	0.1724	0.0426
9	2,980	4,975	1,831	852	5,000	0.9285	0	0	0	0	0	#14	#16	#18	#29
	2,767.05	4,619.49	1,700.16	852	5,000							0.0864	0.0981	0.1588	0.1073
10	3,311	5,494	2,052	1,027	5,840	0.9838	0	0	0	0	0	#14	#16	#18	#29
	3,257.36	5,405.00	2,018.76	1,027	5,840							0.0579	0.1334	0.1171	0.1962
11[a]	3,597	5,494	2,770	1,027	6,600	1.0000	0	0	0	0	0	Itself			
	3,597	5,494	2,770	1,027	6,600										
12	760	850	387	140	986	0.9015	118.41	0	62.97	0	0	#16	#29		
	566.73	766.28	285.91	140	986							0.0769	0.0048		
13	9,100	16,664	7,049	3,000	17,500	1.0000	0	0	0	120	0	#14			
	9,100	16,664	7,049	3,120	17,500							1.0000			
14[a]	9,100	16,664	7,049	3,120	17,500	1.0000	0	0	0	0	0	Itself			
	9,100	16,664	7,049	3,120	17,500										
15[a]	7,280	8,490	2,930	1,476	11,555	1.0000	0	0	0	0	0	Itself			
	7,280	8,490	2,930	1,476	11,555										
16[a]	6,912	9,135	3,395	1,650	12,000	1.0000	0	0	0	0	0	Itself			
	6,912	9,135	3,395	1,650	12,000										
17	3,307	5,224	1,558	848	5,752	0.9898	0	0	0	0	0	#14	#16	#18	#29
	3,273.27	5,170.72	1,542.11	848	5,752							0.0290	0.1439	0.5628	0.0206
18[a]	3,314	5,511	1,321	825	5,773	1.0000	0	0	0	0	0	Itself			
	3,314	5,511	1,321	825	5,773										
19[a]	3,833	3,474	1,197	327	4,920	1.0000	0	0	0	0	0	Itself			
	3,833	3,474	1,197	327	4,920										
20	7,720	9,140	4,135	1,639	11,950	0.9941	768.14	0	742.00	0	0	#15	#16		
	6,906.31	9,086.07	3,368.60	1,639	11,950							0.0366	0.9606		
21	3,783	7,300	3,187	1,210	6,500	0.9105	0	356.15	300.01	0	0	#14	#29		
	3,444.42	6,290.50	2,601.75	1,210	6,500							0.2720	0.1338		
22	751	993	411	180	1,113	0.9496	79.33	0	32.98	0	0	#16	#29		
	633.82	942.95	357.31	180	1,113							0.0607	0.0295		
23	5,461	7,294	2,144	1,142	8,453	0.9628	0	0	0	0	0	#3	#15	#18	#29
	5,257.85	7,022.66	2,064.24	1,142	8,453							0.8459	0.3793	0.5330	0.0080
24	5,203	7,189	2,179	1,150	8,700	0.9986	0	0	0	29.90	0	#15	#16	#18	
	5,195.72	7,178.94	2,175.95	1,179.90	8,700							0.3059	0.1505	0.5819	
25	5,963	9,195	3,187	1,380	9,495	0.8928	0	0	0	41.70	0	#14	#16	#18	
	5,323.77	8,209.30	2,845.35	1,421.70	9,495							0.1430	0.3657	0.4511	
26	7,073	9,814	3,262	1,513	11,000	0.9022	0	0	0	4.49	0	#15	#16	#18	
	6,381.26	8,854.19	2,942.98	1,517.49	11,000							0.0811	0.6206	0.4530	
27	6,733	9,874	3,300	1,604	1,1813	0.9998	0	0	0	86.50	0	#14	#16	#18	
	6,731.92	9,872.42	3,299.47	1,690.50	11,813							0.0671	0.6041	0.5872	
28	5,156	7,306	2,179	1,151	8,495	0.9749	0	0.14	0	11.90	0	#15	#16	#18	
	5,026.58	7,122.48	2,124.31	1,162.90	8,495							0.2253	0.1787	0.6490	
29[a]	7,241	13,133	5,113	2,700	13,000	1.0000	0	0	0	0	0	Itself			
	7,241	13,133	5,113	2,700	13,000										

[a]DEA efficient engines.
[b]For the input–output data of each engine, the first row represents the actual data observed and the row immediately following represents the CCR projected values.
[c]Under this column, "Itself" means that there is no member of peer group for the engine being evaluated except itself.

done in engineering, an ideal (theoretical maximum) device could be added to the group to ensure that all devices would contain the ideal device in their peer group. However, this was not done here because we wanted to emphasize that DEA proceeds with minimal structure and can be used when such additional knowledge or theory is not available.

So far as we are aware, this is a first use of DEA in engineering. See the extensive bibliography in Ref. 2 on the uses of DEA. Subsequent uses of DEA in engineering are quite possible, so future study is warranted that could also lead to new tools and still further extensions.

Appendix B: Envelopment Map and the Related Interpretations

Figure B1 shows an envelopment map for the 29 engines that were evaluated by DEA. The basic idea is to use this matrix arrangement to examine the extent to which the evaluations of each engine reflect interactions with other engines like ones that should be expected to appear in the relative evaluations that are being effected in DEA.

Reference to the total of 4 for engine 3 at the bottom of its column reflects the five check marks that show where this engine was used as part of a reference group to evaluate other engines. Subtracting the one time it was used to evaluate itself produces the net, four

Eng.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	N*
1			✓												✓														✓	3
2			✓												✓														✓	3
3			✓																											0
4			✓												✓			✓											✓	4
5											✓					✓														2
6																✓													✓	2
7															✓	✓		✓												3
8															✓	✓		✓											✓	4
9														✓		✓		✓											✓	4
10														✓		✓		✓											✓	4
11											✓																			0
12																✓													✓	2
13														✓																1
14														✓																0
15															✓															0
16																✓														0
17														✓		✓		✓											✓	4
18																		✓												0
19																			✓											0
20															✓	✓														2
21														✓															✓	2
22																✓													✓	2
23			✓												✓			✓											✓	4
24															✓	✓		✓												3
25														✓		✓		✓												3
26															✓	✓		✓												3
27														✓		✓		✓												3
28															✓	✓		✓												3
29																													✓	0
TN*	0	0	4	0	0	0	0	0	0	0	1	0	0	7	10	15	0	12	0	0	0	0	0	0	0	0	0	0	12	61

TN* ≡ Total number of times that engine *j* (= 1, ..., 29 in column) was used to evaluate *other* engines.
N* ≡ Number of times that *other* engines were used to evaluate engine *i* (= 1, ..., 29 in row).

Fig. B1 Envelopment map for 29 jet engines.

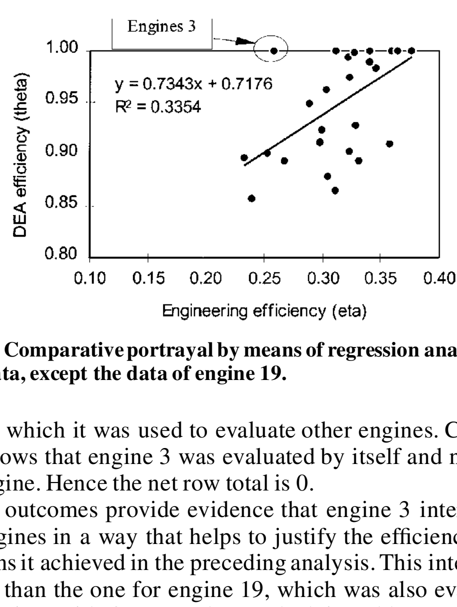

Fig. B2 Comparative portrayal by means of regression analysis for 28 engine data, except the data of engine 19.

times, in which it was used to evaluate other engines. Conversely row 3 shows that engine 3 was evaluated by itself and not by any other engine. Hence the net row total is 0.

These outcomes provide evidence that engine 3 interacts with other engines in a way that helps to justify the efficiency characterizations it achieved in the preceding analysis. This interaction is different than the one for engine 19, which was also evaluated as DEA efficient with $\theta^* = 1$ and zero slack in Table A2. Reference to the net value at the foot of column 19 and on the right for row 19 shows a zero in both cases. Hence engine 19 did not serve as a member of a reference group to evaluate any other engine and was not evaluated by any other engine. It truly seems to be an outlier in the statistical sense or, in DEA terms, it has the property of what is called an inefficient self-evaluator. For a detailed examination of inefficient self-evaluators with their identification and uses in DEA, see Ref. 14.

We now use this information to refine our approach to the treatment of outliers. In the text of the paper, we removed both engines 3 and 19. Here we remove only engine 19 and calculate a new regression to achieve the results shown in Fig. B2. As can be seen in this figure, this regression lies in between the other two regressions in both its slope and intercept values. The results are statistically significantly different from zero, with $R^2 = 0.3354$, $t = 3.622$, and $p = 0.000656$.

Applying the Chow test to Figs. 2b and B2, we obtain $F_{2,51} = 5.65332$ which is highly significant with $p = 0.00606$. We utilized Fig. 2b in reaching our conclusions on the relations between the engineering and the DEA definitions of efficiencies. These conclusions are not changed by moving to Fig. B2, as we now do for the further light it can throw on DEA and its uses.

In moving from Fig. 2a to Fig. 2b we proceeded to remove both engines 3 and 19, as is common in the statistical treatment of outliers. Now we proceed to remove only engine 19. From a statistical standpoint the same results would be secured by removing engine 3. However, as our envelopment map shows, engine 3 is involved in the evaluation of four other engines whereas engine 19 is involved only in its own evaluation. Thus the removal of engine 19 has no further effect. Removal of engine 3, however, can (and generally will) have effects on the efficiency value of other engines. In particular, its removal can effect the efficiency values of engines 1, 2, 4, and 23. This means that the removal of engine 3 will produce a change in the efficiency scores of these other engines unless, in every one of these four cases, there is an alternative optimum that does not involve engine 3 actively as a member of the peer group that forms the basis used in arriving at the performance evaluation.

Thus we see that DEA and its methods, in addition to its uses in engineering can help in statistical analyses of outliers. Other uses in statistics, including uses of DEA to check statistical results, may be found in Ref. 15.

References

[1] Mattingly, J. D., *Elements of Gas Turbine Propulsion*, McGraw–Hill, New York, 1996.

[2] Charnes, A., Cooper, W. W., Lewin, A. Y., and Seiford, L. M., *Data Envelopment Analysis: Theory, Methodology, and Application*, Kluwer Academic, Norwell, MA, 1994, Pt. 1, Chaps. 1–5.

[3] Charnes, A., Cooper, W. W., and Rhodes, E., "Measuring the Efficiency of Decision Making Units," *European Journal of Operational Research*, Vol. 2, No. 6, 1978, pp. 429–444.

[4] Arnold, V., Bardhan, I., Cooper, W. W., and Gallegos, A., "Primal and Dual Optimality in Computer Codes Using Two-Stage Solution Procedures in DEA," *Operations Research: Methods, Models and Applications (A Volume in Honor of G. L. Thompson)*, edited by J. Aranson and S. Zionts, Greenwood, Westport, CT, 1998, Chap. 5, pp. 57–96.

[5] Charnes, A., and Cooper, W. W., "Programming with Linear Fractional Functionals," *Naval Research Logistics Quarterly*, Vol. 9, Nos. 3–4, 1962, pp. 181–185.

[6] Charnes, A., and Cooper, W. W., "Preface to Topics in Data Envelopment Analysis," *Annals of Operations Research*, Vol. 2, 1985, pp. 59–94.

[7] Cooper, W. W., Thompson, R. G., and Thrall, R. M., "Introduction: Extensions and New Developments in DEA," *Annals of Operations Research*, Vol. 66, 1996, pp. 3–45.

[8] Cooper, W. W., Park, K. S., and Pastor, J. T., "RAM: A Range Adjusted Measure of Inefficiency for Use with Additive Models, and Relations to Other Models and Measures in DEA," *Journal of Productivity Analysis*, Vol. 11, No. 1, 1999, pp. 5–42.

[9] Charnes, A., Cooper, W. W., Divine, D., Ruefli, T. W., and Thomas, D., "Comparisons of DEA and Existing Ratio and Regression Systems for Effecting Efficiency Evaluations of Regulated Electric Cooperatives in Texas," *Research in Governmental and Nonprofit Accounting*, Vol. 5, 1989, pp. 187–210.

[10] Bent, R. D., and McKinley, J. L., *Aircraft Powerplants*, 5th ed., Gregg Division, McGraw–Hill, New York, 1985.

[11] Jackson, P., Munson, K., and Taylor, J. W. R., *Jane's All the World's Aircraft*, Butler and Tanner, London, 1995 and 1996.

[12] Charnes, A., Clark, C. T., Cooper, W. W., and Golany, B., "A Developmental Study of Data Envelopment Analysis in Measuring the Efficiency of Maintenance Units in the U.S. Air Forces," *Annals of Operations Research*, Vol. 2, 1985, pp. 75–112.

[13] Charnes, A., Cooper, W. W., and Thrall, R. M., "A Structure for Classifying and Characterizing Efficiency and Inefficiency in Data Envelopment Analysis," *Journal of Productivity Analysis*, Vol. 2, No. 2, 1991, pp. 197–237.

[14] Clarke, R. L., "Effects of Repeated Applications of Data Envelopment Analysis on Efficiency of Air Force Vehicle Maintenance Units in the Tactical Air Command of the U.S. Air Force," Ph.D Dissertation, Graduate School of Business, Univ. of Texas, Austin, TX, 1988; also available from Univ. Microfilms, Inc., Ann Arbor, MI.

[15] Bardhan, I., Cooper, W. W., and Kumbhakar, S. C., "A Simulation Study of Joint Uses of Data Envelopment Analysis and Statistical Regressions for Production Function Estimation and Efficiency Evaluation," *Journal of Productivity Analysis*, Vol. 9, No. 3, 1998, pp. 249–278.